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Weakly open sets in the unit ball of the projective tensor product of Banach spaces ☆

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ABSTRACT

A Banach space is said to have the diameter two property if every non-empty relatively weakly open subset of its unit ball has diameter two. We prove that the projective tensor product of two Banach spaces whose centralizer is infinite-dimensional has the diameter two property. The same statement also holds for $X \widehat{\otimes}_\pi Y$ if the centralizer of X is infinite-dimensional and the unit sphere of Y^* contains an element of numerical index one. We provide examples of classical Banach spaces satisfying the assumptions of the results. If K is any infinite compact Hausdorff topological space, then $\mathcal{C}(K) \widehat{\otimes}_\pi Y$ has the diameter two property for any nonzero Banach space Y. We also provide a result on the diameter two property for the injective tensor product.

1. Introduction

Nygaard and Werner showed that some of the classical Banach spaces without the Radon–Nikodým property actually fail much weaker requirements. Indeed they proved that for any infinite-dimensional uniform algebra, every non-empty relatively weakly open subset of its closed unit ball has diameter equal to two [26]. If a Banach space satisfies the above condition, we will say that it has the *diameter two property* [10]. As a consequence of the mentioned result, every infinite-dimensional real or complex $\mathcal{C}(K)$ satisfies the diameter two property. The result for $\mathcal{C}(K)$ was extended to real JB^*-triples (in the sense of [18]) whose Banach space is not reflexive by Becerra, López, Peralta and Rodríguez [8]. Hence every infinite-dimensional C^*-algebra satisfies the diameter two property (see also [9]).

Becerra and López proved that for every atomless measure μ and for every compact Hausdorff topological space K, the spaces $L_1(\mu, X)$ and $\mathcal{C}(K, X)$ have the diameter two property for every nonzero Banach space X [7]. López obtained positive results for L-embedded and M-embedded Banach spaces under some additional assumptions [23].

Recently the results of [9,8,7] have been generalized and unified in [10] by proving that every Banach spaces whose centralizer is infinite-dimensional satisfies the diameter two property.

It is also known that every Banach space with the Daugavet property has the diameter two property [28]. However, there are spaces without the Daugavet property that enjoy some of its consequences. For instance, in [4] it was proved that the interpolation spaces $L_1 \cap L_\infty$ and $L_1 + L_\infty$ satisfy that every slice of the unit ball have diameter two. For the symmetric projective tensor of some classical Banach spaces some results along the same line can be found in [1–3].

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If a Banach space satisfies that every slice of the unit ball has diameter two, then it is immediate that the same property is satisfied by its projective tensor product with any other nonzero Banach space. For the diameter two property, it is not clear the behavior. Now we will describe briefly the results contained in this paper.

As a consequence of the main result of Section 2, the complete projective tensor product of two Banach spaces whose centralizers are infinite-dimensional, has the diameter two property. The previous result can be applied, for instance, to every infinite-dimensional C^*-algebra, or to any space $\mathcal{C}(K,X)$, for every infinite compact Hausdorff space K and for any nonzero Banach space X. In Section 3 we obtain a result along the same line assuming that the centralizer of one of the Banach spaces is infinite-dimensional and the unit sphere of the dual of the other contains an element of numerical index one. In order to show this statement, we need results on the numerical index that are interesting by themselves. The class of the spaces satisfying the assumption on the numerical index contains the so-called CL-spaces. For instance, the spaces $L_1(\mu)$ and $\mathcal{C}(K)$ are CL-spaces. For the case of $\mathcal{C}(K)$ spaces we obtain a refinement in Section 4. The projective tensor product of every infinite-dimensional $\mathcal{C}(K)$ and any nonzero Banach space satisfies the diameter two property. Finally last section contains one result stating the diameter two property for the injective tensor product under certain assumptions on the Banach spaces. We do not know in general if the projective tensor product of a Banach space with the diameter two property and any other non-trivial Banach space also satisfies the diameter two property.

Throughout the paper, X will be a Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). As usual, S_X, B_X and X^* will denote the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X.

2. A result for the projective tensor product

We recall that a *function module* is (the third coordinate of) a triple $(K,(X_t)_{t\in K},X)$, where K is a non-empty compact Hausdorff topological space (called the base space), $(X_t)_{t\in K}$ a family of Banach spaces, and X a closed $\mathcal{C}(K)$-submodule of the $\mathcal{C}(K)$-module $\prod_{t\in K}^{\infty} X_t$ (the ℓ_∞-sum of the spaces X_t) such that the following conditions are satisfied:

(1) For every $x\in X$, the function $t\to \|x(t)\|$ from K to $\mathbb{R}$ is upper semi-continuous.
(2) For every $t\in K$, we have $X_t=\{x(t)\colon x\in X\}$.
(3) The set $\{t\in K\colon X_t\neq 0\}$ is dense in K.

We follow the notation of [11], where the basic results on function modules can be found.

Lemma 2.1. *(See [10, Lemma 2.1].) Let $(K,(X_t)_{t\in K},X)$ be a function module, and let x be an extreme point of B_X. Then, for every $t\in K$ we have $\|x(t)\|=1$.*

Let X be a Banach space over $\mathbb{K}$ and $\mathcal{L}(X)$ the space of all bounded and linear operators on X. By a *multiplier* on X we mean an element $T\in\mathcal{L}(X)$ such that every extreme point of B_{X^*} becomes an eigenvector for T^*. Thus, given a multiplier T on X, and an extreme point p of B_{X^*}, there exists a unique number $a_T(p)$ satisfying $T^*(p)=a_T(p)p$. The *centralizer* of X (denoted by $Z(X)$) is defined as the set of those multipliers T on X such that there exists a multiplier S on X satisfying $a_S(p)=\overline{a_T(p)}$ for every extreme point p of B_{X^*}. Thus, if $\mathbb{K}=\mathbb{R}$, then $Z(X)$ coincides with the set of all multipliers on X. In all cases, $Z(X)$ is a closed subalgebra of $\mathcal{L}(X)$ isometrically isomorphic to $\mathcal{C}(K_X)$, for some compact Hausdorff topological space K_X (see [11, Proposition 3.10]). Moreover X can be seen as a function module whose base space is precisely K_X, and such that the elements of $Z(X)$ are precisely the operators of multiplication by the elements of $\mathcal{C}(K_X)$ (see [11, Theorem 4.14]).

If X and Y are Banach spaces over the same scalar field ($\mathbb{K}$), we will denote by $\mathcal{B}(X\times Y)$ the space of bounded bilinear forms on $X\times Y$. We recall that *the projective tensor product* of X and Y, denoted by $X\widehat{\otimes}_\pi Y$, is the completion of $X\otimes Y$ under the norm given by

$$\|u\|:=\inf\left\{\sum_{i=1}^n\|x_i\|\|y_i\|\colon u=\sum_{i=1}^n x_i\otimes y_i,\ n\in\mathbb{N},\ x_i\in X,\ y_i\in Y,\ \forall 1\leqslant i\leqslant n\right\}.$$

We recall that the space $\mathcal{B}(X\times Y)$ is linearly isometric to the topological dual of $X\widehat{\otimes}_\pi Y$. Under this identification, for every $A\in\mathcal{B}(X\times Y)$, we will denote by $\bar{A}$ the corresponding linear functional on $X\widehat{\otimes}_\pi Y$. It is satisfied that

$$\bar{A}(x\otimes y)=A(x,y)\quad \forall(x,y)\in X\times Y.$$

Lemma 2.2. *Let X and Y be Banach spaces and assume that B_X contains some extreme point and $Z(X)$ is infinite-dimensional. Let W be an open set in $(B_{X\widehat{\otimes}_\pi Y},w)$ and $z_0\in W$ such that*

$$z_0=\sum_{j=1}^k\alpha_j x_j\otimes y_j,$$

where $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$, $\sum_{j=1}^{k} \alpha_j = 1$, $x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \ldots, k$. Then there are elements $\varphi \in B_{X^*}$ and u_j, v_j in B_X for $1 \leqslant j \leqslant k$ such that

$$\sum_{j=1}^{k} \alpha_j u_j \otimes y_j, \qquad \sum_{j=1}^{k} \alpha_j v_j \otimes y_j \in W$$

and $\varphi(u_j) = 1 = -\varphi(v_j)$ for all $j = 1, \ldots, k$.

Proof. By a previous remark, we can assume that X is a function module with base space equal to some compact $K := K_X$, and such that $Z(X)$ coincide with the set of operators of multiplication by elements of $\mathcal{C}(K)$. Since $Z(X)$ is infinite-dimensional K is infinite. Hence there is a sequence $\{O_n\}$ of non-empty pair-wise disjoint open subsets of K. For $n \in \mathbb{N}$, take $t_n \in O_n$, and apply Urysohn's Lemma to pick f_n in $\mathcal{C}(K)$ with $0 \leqslant f_n \leqslant 1$, $f_n(t_n) = 1$ and $f_n(t) = 0$ whenever $t \in K \setminus O_n$. Since the bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to 0, it converges weakly to 0 in $\mathcal{C}(K)$, and hence $\{f_n x\}_{n\in\mathbb{N}}$ converges weakly to zero and $\{(1-f_n)x\}_{n\in\mathbb{N}}$ converges weakly to x in X for every element x in X. By the assumption there is an extreme point p of B_X. For each $j = 1, \ldots, k$, we define the sequences $\{u_n^j\}$ and $\{v_n^j\}$ in B_X by

$$u_n^j = (1-f_n)x_j + f_n p,$$

and

$$v_n^j = (1-f_n)x_j - f_n p.$$

Let us notice that for every $s \in K$, one has

$$\bigl(1 - f_n(s)\bigr)\bigl\|x_j(s)\bigr\| + f_n(s)\bigl\|p(s)\bigr\| \leqslant 1,$$

hence $\|u_n^j\| \leqslant 1$ and $\|v_n^j\| \leqslant 1$.

Since the sequences $\{u_n^j\}_n$ and $\{v_n^j\}_n$ converge weakly to x_j for all $j = 1, \ldots, k$, we deduce that $\{u_n^j \otimes y_j\}$ and $\{v_n^j \otimes y_j\}$ converge weakly to $x_j \otimes y_j$ in $B_{X\widehat{\otimes}_\pi Y}$, for all $j = 1, \ldots, k$. Then there exists n such that

$$\sum_{j=1}^{k} \alpha_j u_n^j \otimes y_j, \qquad \sum_{j=1}^{k} \alpha_j v_n^j \otimes y_j \in W,$$

we define $u_j := u_n^j$ and $v_j := v_n^j$ for all $j = 1, \ldots, k$. We know that there are elements $t_n \in K$ satisfying $f_n(t_n) = 1$. Since p is an extreme point of B_X, by Lemma 2.1, $\|p(t)\| = 1$ for every $t \in K$. Let $\varphi_{t_n} \in B_{(X(t_n))^*}$, such that $\varphi_{t_n}(p(t_n)) = 1$. So the functional defined by $\varphi(x) := \varphi_{t_n}(x(t_n))$ $(x \in X)$ belongs to B_{X^*}. We have that $\varphi(u_j) = 1 = -\varphi(v_j)$ for every $j = 1, \ldots, k$. $\square$

Every (bounded) bilinear form $A : X \times Y \to \mathbb{K}$ can be identified with an operator $T : X \to Y^*$ by the formula $T(x)(y) = A(x, y)$ for $(x, y) \in X \times Y$. We will denote by $\widehat{A}^{(2}$ the bilinear form on $X^{**} \times Y$ associated to the w^*-continuous operator $S := (J_Y)^* \circ T^{**}$, where $J_Y : Y \to Y^{**}$ is the canonical injection of Y in its bidual. Since T^{**} is an extension of T, then $\widehat{A}^{(2}$ is an extension of A. Indeed $\widehat{A}^{(2}$ is the restriction to $X^{**} \times Y$ of the Arens extension of A. Since S is w^*-continuous, then $\widehat{A}^{(2}$ satisfies

$$\widehat{A}^{(2}\bigl(x^{**}, y\bigr) = \lim_{\alpha} A(x_\alpha, y) \quad \forall \bigl(x^{**}, y\bigr) \in X^{**} \times Y, \tag{2.1}$$

for every net (x_α) in X that converges to x^{**} in the w^*-topology of X^{**}.

Lemma 2.3. *Let X and Y be Banach spaces and assume that $Z(X)$ is infinite-dimensional. Let W be an open set in $(B_{X\widehat{\otimes}_\pi Y}, w)$ and $z_0 \in W$ that can be written as*

$$z_0 = \sum_{j=1}^{k} \alpha_j x_j \otimes y_j,$$

where $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$, $\sum_{j=1}^{k} \alpha_j = 1$, $x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \ldots, k$. Then for every $\varepsilon > 0$, there exists u_j, v_j in B_X and $\varphi \in B_{X^*}$ satisfying

$$\sum_{j=1}^{k} \alpha_j u_j \otimes y_j, \qquad \sum_{j=1}^{k} \alpha_j v_j \otimes y_j \in W$$

and $|\varphi(u_j) - 1| < \varepsilon$ and $|\varphi(v_j) + 1| < \varepsilon$ for every $j = 1, \ldots, k$.

Proof. Since $z_0 \in W$, we can assume that there are $\eta > 0$ and $A_1, \dots, A_m$ in $\mathcal{B}(X \times Y)$ such that

$$W := \{z \in B_{X \widehat{\otimes}_\pi Y}\colon \left|\bar{A}_i(z) - \bar{A}_i(z_0)\right| < \eta,\ \forall 1 \leqslant i \leqslant m\}.$$

We write, for each $i \in \{1, \dots, m\}$, $B_i := \widehat{A_i}^{(2}$, the extension of A_i to $X^{**} \times Y$ described above. Now, we consider the weakly open set of $B_{X^{**} \widehat{\otimes}_\pi Y}$ given by

$$\widehat{W} := \{\widehat{z} \in B_{X^{**} \widehat{\otimes}_\pi Y}\colon \left|\bar{B}_i(\widehat{z}) - \bar{B}_i(z_0)\right| < \eta,\ \forall 1 \leqslant i \leqslant m\}.$$

It is clear that $W \subset \widehat{W}$, so $z_0 \in \widehat{W}$. By assumption, $Z(X)$ is infinite-dimensional, so $Z(X^{**})$ is also infinite-dimensional in view of [17, Corollary I.3.15]. Since $B_{X^{**}}$ has extreme points, we can apply Lemma 2.2 to the element $z_0 \in \widehat{W}$. We have that $\widehat{W}$ contains elements $\widehat{z}_1, \widehat{z}_2$ that can be expressed as

$$\widehat{z}_1 = \sum_{j=1}^{k} \alpha_j u_j^{**} \otimes y_j,$$

and

$$\widehat{z}_2 = \sum_{j=1}^{k} \alpha_j v_j^{**} \otimes y_j,$$

where $u_j^{**}, v_j^{**} \in S_{X^{**}}$, and there exist $\varphi \in B_{X^{***}}$ such that $\varphi(u_j^{**}) = 1 = -\varphi(v_j^{**})$ for all $j = 1, \dots, k$. Given $\varepsilon > 0$, since B_{X^*} is w^*-dense in $B_{X^{***}}$, we can assume that $\varphi \in B_{X^*}$ and $|\varphi(u_j^{**}) - 1| < \varepsilon$ and $|\varphi(v_j^{**}) + 1| < \varepsilon$ for all $j = 1, \dots, k$.

Since S_X is w^*-dense in $S_{X^{**}}$ and each $\widehat{A_i}^{(2}$ is w^*-continuous on the first variable for every $1 \leqslant i \leqslant m$, there are $u_j, v_j \in S_X$ such that $|\varphi(u_j) - 1| < \varepsilon$ and $|\varphi(v_j) + 1| < \varepsilon$ for all $j \in \{1, \dots, k\}$ and the elements

$$z_1 := \sum_{j=1}^{k} \alpha_j u_j \otimes y_j, \qquad z_2 = \sum_{j=1}^{k} \alpha_j v_j \otimes y_j$$

satisfy that $z_1, z_2 \in W$. $\square$

Given a Banach space X, we consider the increasing sequence of its even duals

$$X \subseteq X^{**} \subseteq X^{(4} \subseteq \cdots \subseteq X^{(2n} \subseteq \cdots,$$

and we define $X^{(\infty}$ as the completion of the normed space $\bigcup_{n=0}^{\infty} X^{(2n}$.

Proposition 2.4. *(See [2, Proposition 3.1].) Let X be a Banach space. Then $B_{(X^*)^{(\infty}}$ is w^*-dense in $B_{(X^{(\infty)})^*}$.*

For every Banach spaces X and Y, we will show that there is a natural embedding $A \to \widetilde{A}$ from $\mathcal{B}(X \times Y)$ to $\mathcal{B}(X^{(\infty} \times Y)$. Let us recall that we denoted by $\widehat{A}^{(2}$ the extension of A to $X^{**} \times Y$ that we described before. We know that this canonical extension satisfies $\|\widehat{A}^{(2}\| = \|A\|$. We denote by $\widehat{A}^{(2n}$ the extension of $\widehat{A}^{(2n-2}$ to $X^{(2n} \times Y$ defined in (2.1). We have that $\|\widehat{A}^{(2n}\| = \|A\|$ for all $n \in \mathbb{N}$. Indeed, let A be in $\mathcal{B}(X \times Y)$. Given $\alpha \in \bigcup_{n=0}^{\infty} X^{(2n} \times Y$, there exists $m \in \mathbb{N}$ such that α belongs to $X^{(2m} \times Y$, allowing us to consider the element $\widehat{A}^{(2m}(\alpha)$, which is well defined. In this way we are provided with a natural extension of A to $\bigcup_{n=0}^{\infty} X^{(2n} \times Y$, which extends uniquely by continuity to $X^{(\infty} \times Y$, giving rise to an element $\widetilde{A}$ of $\mathcal{B}(X^{(\infty} \times Y)$.

In this way we have the following chain of embeddings

$$\mathcal{B}(X \times Y) \hookrightarrow \mathcal{B}(X^{**} \times Y) \hookrightarrow \mathcal{B}(X^{(4} \times Y) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(X^{(2n} \times Y) \hookrightarrow \cdots,$$

where each arrow means the corresponding extension.

Hence we can complete the above chain as follows

$$\mathcal{B}(X \times Y) \hookrightarrow \mathcal{B}(X^{**} \times Y) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(X^{(2n} \times Y) \hookrightarrow \cdots \hookrightarrow \mathcal{B}(X^{(\infty} \times Y),$$

and the embedding $A \to \widetilde{A}$ from $\mathcal{B}(X \times Y)$ to $\mathcal{B}(X^{(\infty} \times Y)$ is an isometry.

Lemma 2.5. *Let X and Y be Banach spaces and assume that $Z(X^{(\infty)})$ is infinite-dimensional. Let W be an open set in $(B_{X \widehat{\otimes}_\pi Y}, w)$ and $z_0 \in W$ such that*

$$z_0 = \sum_{j=1}^{k} \alpha_j x_j \otimes y_j,$$

where $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$, $\sum_{j=1}^{k} \alpha_j = 1$, $x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \dots, k$. Then for every $\varepsilon > 0$, there exists u_j, v_j in B_X and $\varphi \in B_{X^*}$ such that

$$\sum_{j=1}^{k} \alpha_j u_j \otimes y_j, \qquad \sum_{j=1}^{k} \alpha_j v_j \otimes y_j \in W$$

and $|\varphi(u_j) - 1| < \varepsilon$ and $|\varphi(v_j) + 1| < \varepsilon$ for every $1 \leqslant j \leqslant k$.

Proof. Since $z_0 \in W$, we can assume that there is $\eta > 0$ and $A_1, \dots, A_m$ in $\mathcal{B}(X \times Y)$ such that

$$W := \bigl\{ z \in B_{X \widehat{\otimes}_\pi Y} \colon \bigl| \bar{A}_i(z) - \bar{A}_i(z_0) \bigr| < \eta,\ \forall 1 \leqslant i \leqslant m \bigr\}.$$

Let us consider, for each $i \in \{1, \dots, m\}$, the extension $\widetilde{A_i}$ of A_i to $X^{(\infty} \times Y$. We denote by L_i the linear functional on $X^{(\infty} \widehat{\otimes}_\pi Y$ associated to the bilinear form $\widetilde{A_i}$ for $1 \leqslant i \leqslant m$. Now we define the weakly open set in the unit ball of $X^{(\infty} \widehat{\otimes}_\pi Y$ by

$$\widehat{W} := \bigl\{ \widehat{z} \in B_{X^{(\infty} \widehat{\otimes}_\pi Y} \colon \bigl| L_i(\widehat{z}) - L_i(z_0) \bigr| < \eta,\ \forall 1 \leqslant i \leqslant m \bigr\}.$$

We know that $z_0 \in \widehat{W}$. By assumption, $Z(X^{(\infty)})$ is infinite-dimensional, so we can apply Lemma 2.3 to $z_0 \in \widehat{W}$. We obtain that $\widehat{W}$ contain elements $\widehat{z}_1, \widehat{z}_2$ that can be expressed as

$$\widehat{z}_1 = \sum_{j=1}^{k} \alpha_j u_j^{(\infty} \otimes y_j,$$

and

$$\widehat{z}_2 = \sum_{j=1}^{k} \alpha_j v_j^{(\infty} \otimes y_j,$$

where $u_j^{(\infty}, v_j^{(\infty} \in S_{X^{(\infty}}$, and there exists $\varphi \in B_{(X^{(\infty)})^*}$ such that $|\varphi(u_j^{(\infty)}) - 1| < \varepsilon$ and $|\varphi(v_j^{(\infty)}) + 1| < \varepsilon$ for all $j = 1, \dots, k$. By Proposition 2.4 we know that $B_{(X^*)^{(\infty}}$ is w^*-dense in $B_{(X^{(\infty)})^*}$, so by using the definition of $(X^*)^{(\infty}$ we can assume that there exists $p \in \mathbb{N}$ such that $\varphi \in B_{(X^*)^{(2p}}$. Now, by the definition of $X^{(\infty}$, we can assume that there exists $q \in \mathbb{N}$ such that $u_j^{(\infty}, v_j^{(\infty} \in B_{X^{(2q}}$, for all $j = 1, \dots, k$. This implies that there exists $n \in \mathbb{N}$ such that $u_j^{(\infty}, v_j^{(\infty} \in B_{X^{(2n}}$, for all $j = 1, \dots, k$, and $\varphi \in B_{(X^*)^{(2n}}$. If we proceed as in the last part of the proof of Lemma 2.3, after a finite number of steps, we conclude the proof. $\square$

Theorem 2.6. *Let X and Y be Banach spaces such that $Z(X^{(\infty)})$ and $Z(Y^{(\infty)})$ are infinite-dimensional. Then the space $X \widehat{\otimes}_\pi Y$ has the diameter two property.*

Proof. Let W be a non-empty open set in $(B_{X \widehat{\otimes}_\pi Y}, w)$. We can clearly assume that there is $\eta > 0$, $A_1, \dots, A_m$ in $\mathcal{B}(X \times Y)$, and $z_0 \in B_{X \widehat{\otimes}_\pi Y}$ such that

$$W := \bigl\{ z \in B_{X \widehat{\otimes}_\pi Y} \colon \bigl| \bar{A}_i(z) - \bar{A}_i(z_0) \bigr| < \eta,\ \forall 1 \leqslant i \leqslant m \bigr\}.$$

Since every weakly open set is norm open set, we suppose that

$$z_0 = \sum_{j=1}^{k} \alpha_j x_j \otimes y_j,$$

where $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$, $\sum_{j=1}^{k} \alpha_j = 1$, $x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \dots, k$. Given $\varepsilon > 0$, by applying Lemma 2.5 to the Banach space Y, there are elements w_j in B_Y for $1 \leqslant j \leqslant k$ and $\psi \in B_{Y^*}$ such that

$$\sum_{j=1}^{k} \alpha_j x_j \otimes w_j \in W$$

and $|\psi(w_j) - 1| < \varepsilon$ for all $j = 1, \dots, k$. If we apply Lemma 2.5 to X and $\sum_{j=1}^{k} \alpha_j x_j \otimes w_j$, there are u_j, v_j in B_X $(1 \leqslant j \leqslant k)$ and $\varphi \in B_{X^*}$ such that

$$z_1 = \sum_{j=1}^{k} \alpha_j u_j \otimes w_j, \qquad z_2 = \sum_{j=1}^{k} \alpha_j v_j \otimes w_j \in W$$

and $|\varphi(u_j) - 1| < \varepsilon$ and $|\varphi(v_j) + 1| < \varepsilon$ for all $j = 1, \dots, k$. We consider the bilinear map $\phi : X \times Y \to \mathbb{K}$ given by $\phi(x, y) := \varphi(x)\psi(y)$, which is bounded and satisfies $\|\phi\| \leqslant 1$. Then

$$\begin{aligned} \|z_1 - z_2\| \geqslant \big|\bar{\phi}(z_1 - z_2)\big| &= \left|\sum_{j=1}^{k} \alpha_j \big(\varphi(u_j) - \varphi(v_j)\big)\psi(w_j)\right| \\ &\geqslant \sum_{j=1}^{k} \alpha_j (2 - 2\varepsilon)(1 - \varepsilon) = (2 - 2\varepsilon)(1 - \varepsilon). \end{aligned}$$

We conclude that

$$(2 - 2\varepsilon)(1 - \varepsilon) \leqslant \|z_1 - z_2\| \leqslant \operatorname{diam} W \leqslant 2,$$

for every $\varepsilon > 0$. Hence $\operatorname{diam} W = 2$ as we wanted to show. □

Now we will provide examples of spaces where the previous results can be applied. Given a Banach space X, there exists a natural embedding $T \to \widetilde{T}$ from $\mathcal{L}(X)$ to $\mathcal{L}(X^{(\infty})$. Indeed, let T be in $\mathcal{L}(X)$. Given $\alpha \in \bigcup_{n=0}^{\infty} X^{(2n}$, there exists $m \in \mathbb{N}$ such that α belongs to $X^{(2m}$, allowing us to consider the element $T^{(2m}(\alpha)$ of $\bigcup_{n=0}^{\infty} X^{(2n}$, which does not depend on m. In this way we are provided with a natural extension of T to $\bigcup_{n=0}^{\infty} X^{(2n}$, which extends uniquely by continuity to $X^{(\infty}$, giving rise to an element $\widetilde{T}$ of $\mathcal{L}(X^{(\infty})$. It is known that for every T in $Z(X)$, T^{**} lies in $Z(X^{**})$ [17, Corollary I.3.15]. Hence we already are aware of the chain of embeddings

$$Z(X) \hookrightarrow Z\big(X^{**}\big) \hookrightarrow Z\big(X^{****}\big) \hookrightarrow \cdots \hookrightarrow Z\big(X^{(2n}\big) \hookrightarrow \cdots,$$

where each arrow means double transposition.

Indeed, it is known that the image of $Z(X)$ under this embedding is contained in $Z(X^{(\infty})$ (see [10, Proposition 4.3]). Hence we can complete the above chain as follows

$$Z(X) \hookrightarrow Z\big(X^{**}\big) \hookrightarrow Z\big(X^{****}\big) \hookrightarrow \cdots \hookrightarrow Z\big(X^{(2n}\big) \hookrightarrow \cdots \hookrightarrow Z\big(X^{(\infty}\big).$$

For a Banach space X, an *L-projection* on X is a (linear) projection $P : X \to X$ satisfying $\|x\| = \|P(x)\| + \|x - P(x)\|$ for every $x \in X$. In such a case, we will say that the subspace $P(X)$ is an *L-summand* of X. Let us notice that the composition of two L-projections on X is an L-projection [11, Proposition 1.7], so the closed linear subspace of $\mathcal{L}(X)$ generated by all L-projections on X is a subalgebra of $\mathcal{L}(X)$, the space of all bounded and linear operators on X. This algebra, denoted by $C(X)$, is called the *Cunningham algebra* of X. It is known that $C(X)$ is linearly isometric to $Z(X^*)$ (see [11, Theorems 5.7 and 5.9]).

The following Banach spaces X satisfy that $\sup\{\dim Z(X^{(2n}): n \in \mathbb{N}\} = \infty$, so $Z(X^{(\infty})$ is infinite-dimensional:

(1) Every non-reflexive Banach space X such that X^* is L-embedded [2, Proposition 3.3]. For instance, an infinite-dimensional predual of an L_1-space or a (real or complex) infinite-dimensional C^*-algebra belongs to this class.
(2) The space $\mathcal{C}(K, (X, \tau))$ where K is an infinite compact topological space, X is a non-null Banach space and τ is a topology such that the weak topology is contained in τ and the norm topology is finer than τ [10, Proposition 3.2].
(3) $\mathcal{L}(X, Y)$ (the space of all bounded and linear operators from X to Y) for every Banach spaces X, Y such that either $C(X)$ is infinite-dimensional or $Z(Y)$ is infinite-dimensional (see [17, Lemma VI.1.1]). For instance, any infinite-dimensional space $L_1(\mu)$ satisfies that its Cunningham algebra is infinite-dimensional.

Under some isomorphic condition, Banach spaces have an equivalent norm satisfying the assumption of Theorem 2.6. The diameter two property for an equivalent norm under the next assumption was previously obtained in [23, Proposition 2.6] by the same procedure.

Proposition 2.7. *Let X be a Banach space containing an isomorphic copy of c_0. Then there is an equivalent norm on X for which*

$$X^{**} = \ell_\infty \oplus_\infty N.$$

Hence $Z(X^{**})$ is infinite-dimensional.

Proof. Under the current assumption there is an equivalent norm on X for which X contains an isometric copy of c_0. By using [17, Proposition II.2.10], there is another equivalent norm on X that contains an isometric copy of c_0 and such that c_0 is M-ideal of X, that is, $X^{**} = \ell_\infty \oplus_\infty N$ for some subspace N of X^{**}. □

3. Quantifying the notion of an w^*-unitary

By a CS-closed set in a Banach space X we mean a subset S of X such that, whenever $\sum_{n=1}^{\infty}\alpha_n s_n = x \in X$, with $s_n \in S$, $\alpha_n \geqslant 0$, and $\sum_{n=1}^{\infty}\alpha_n = 1$, we have $x \in S$. The celebrated Banach principle, that quasi-open continuous linear mappings between Banach spaces are open, is codified in Jameson's book [19] as follows.

Lemma 3.1. *(See [19, Theorem 22.4].) In a Banach space, a CS-closed set and its closure have the same interior.*

For a subset S of a Banach space, $\mathrm{co}(S)$, $|\mathrm{co}|(S)$, and $\overline{|\mathrm{co}|}(S)$ will denote the convex, absolutely convex, and closed absolutely convex hull of S, respectively.

Corollary 3.2. *Let S be a CS-closed set in a Banach space X. Then $|\mathrm{co}|(S)$ and $\overline{|\mathrm{co}|}(S)$ have the same interior in X.*

Proof. If X is real, then $|\mathrm{co}|(S) = \mathrm{co}(S \cup -S)$ is a CS-closed set (by [19, 22.2 and 22.3]), and the result follows from Lemma 3.1. Assume that X is complex. Let $\varepsilon > 0$, and take $n \in \mathbb{N}$ such that $B_{\mathbb{C}} \subseteq (1+\varepsilon)\,\mathrm{co}(\{z_1,\dots,z_n\})$, where $z_1,\dots,z_n$ are the n-th roots of 1 in $\mathbb{C}$. Then we have

$$|\mathrm{co}|(S) = \mathrm{co}(B_{\mathbb{C}}S) \subseteq (1+\varepsilon)\,\mathrm{co}\left(\bigcup_{i=1}^{n} z_i S\right) \subseteq (1+\varepsilon)|\mathrm{co}|(S).$$

By keeping in mind that $\mathrm{co}(\bigcup_{i=1}^{n} z_i S)$ is a CS-closed set, and applying Lemma 3.1, we deduce that $T \subseteq (1+\varepsilon)|\mathrm{co}|(S)$, where T stands for the interior of $\overline{|\mathrm{co}|}(S)$. Therefore, since T is open, we have $T \subseteq \bigcup_{\varepsilon>0}\frac{1}{1+\varepsilon}T \subseteq |\mathrm{co}|(S)$. □

Let X be a Banach space, and let u be a norm-one element in X. We put

$$D(X,u) := \{f \in B_{X^*}\colon\ f(u) = 1\}.$$

Now, assume that X has a (complete) predual X_*, and put

$$D^{w^*}(X,u) := D(X,u) \cap X_*.$$

If $D^{w^*}(X,u) = \emptyset$, then we define $n^{w^*}(X,u) := 0$. Otherwise, we define $n^{w^*}(X,u)$ as the largest non-negative real number k satisfying

$$k\|x\| \leqslant v^{w^*}(x) := \sup\{|f(x)|\colon\ f \in D^{w^*}(X,u)\}$$

for every $x \in X$. We say that u is an w^*-unitary element of X if the linear hull of $D^{w^*}(X,u)$ equals the whole space X_*.

Proposition 3.3. *Let u be a norm-one element in a dual Banach space X. Then u is w^*-unitary in X if and only if $n^{w^*}(X,u) > 0$. Moreover, we have*

$$n^{w^*}(X,u)\,\mathrm{int}(B_{X_*}) \subseteq |\mathrm{co}|\big(D^{w^*}(X,u)\big). \tag{3.1}$$

Proof. Assume that u is w^*-unitary. Then $\overline{|\mathrm{co}|}(D^{w^*}(X,u))$ is a barrel in X_*. Since barrels in a Banach space are neighborhoods of zero, there exists $k > 0$ such that $kB_{X_*} \subseteq \overline{|\mathrm{co}|}(D^{w^*}(X,u))$. This implies that $k\|x\| \leqslant v^{w^*}(x)$ for every $x \in X$, and hence that $n^{w^*}(X,u) > 0$.

Now we can clearly assume that $n^{w^*}(X,u) > 0$. Then, in the duality (X, X_*), the set

$$B := \{x \in X\colon\ v^{w^*}(x) \leqslant 1\}$$

is the absolute polar of $D^{w^*}(X,u)$, and the inclusion $B \subseteq \frac{1}{n^{w^*}(X,u)}B_X$ holds. It follows from the bipolar theorem that $n^{w^*}(X,u)B_{X_*} \subseteq \overline{|\mathrm{co}|}(D^{w^*}(X,u))$. By applying Corollary 3.2, the inclusion (3.1) follows. Clearly, that inclusion implies that u is w^*-unitary. □

The first paragraph in the above proof is taken from the proof of [6, Corollary 3.2].

Now, let X be an arbitrary Banach space, and let u be a norm-one element in X. We define $n(X,u)$ as the largest non-negative real number k satisfying

$$k\|x\| \leqslant v(x) := \sup\{|f(x)|\colon\ f \in D(X,u)\}$$

for every $x \in X$, and we say that u is an unitary element of X if the linear hull of $D(X,u)$ equals the whole space X^*.

Let σ be a topology on a set E, let τ be a vector space topology on a vector space Y over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), let f be a function from E to 2^Y (empty values for f are allowed), and let u be in E. We say that f is σ-τ upper semi-continuous (in short, σ-τ usc) at u if, for every τ-neighborhood V of zero in Y, there exists a σ-neighborhood U of u in E such that $f(x) \subseteq f(u) + V$ whenever x lies in U.

Now, let X be a dual Banach space. We define the pre-duality mapping of X as the function $x \to D^{w^*}(X, x)$ from the unit sphere of X to 2^{X_*}. Let u be a norm-one element in X. Since, clearly, $n^{w^*}(X, u) \leqslant n(X, u)$, it follows from Proposition 3.3 and [25, Theorem 3.1] that, if u is w^*-unitary, then u is unitary [6, Corollary 3.2]. Under the requirement that the pre-duality mapping of X is norm-weak usc at u, the converse is also true. This result is due to G. Godefroy and T.S.S.R.K. Rao [16, Proposition 2.2] in the real case, and its proof is clarified and adapted to the complex case in [27]. Now we are ready to formulate and prove a quantification of the result just quoted.

Theorem 3.4. *Let u be an unitary element in a dual Banach space X, and assume that the pre-duality mapping of X is norm-weak usc at u. Then u is w^*-unitary. More precisely, we have $n(X, u) = n^{w^*}(X, u) > 0$ and*

$$n(X, u)\,\mathrm{int}(B_{X_*}) \subseteq |\mathrm{co}|\big(D^{w^*}(X, u)\big).$$

Proof. According to [15, Lemma 2.2], the assumption that the pre-duality mapping of X is norm-weak usc at u is equivalent to the fact that $D^{w^*}(X, u)$ is w^*-dense in $D(X, u)$. Therefore we have $v(x) = v^{w^*}(x)$ for every $x \in X$, and, consequently, the equality $n(X, u) = n^{w^*}(X, u)$ holds. Now apply Proposition 3.3. $\square$

Now, let X be an arbitrary Banach space. The duality mapping of X is defined as the function $x \to D(X, x)$ from the unit sphere of X to 2^{X^*}. Let u be a norm-one element in X. If $n(X, u) = 1$, then u is both an unitary element of X (by [25, Theorem 3.1]) and a point of norm–norm upper semi-continuity of the duality mapping of X [5, Corollary 5.9]. On the other hand, in the case that X is in fact a dual Banach space, the mere norm-weak upper semi-continuity of the duality mapping of X at u implies the norm-weak upper semi-continuity of the pre-duality mapping of X at u [15, Theorem 2.3]. Therefore, by invoking Theorem 3.4, we get the following.

Corollary 3.5. *Let X be a dual Banach space, and let u be a norm-one element in X such that $n(X, u) = 1$. Then we have*

$$\mathrm{int}(B_{X_*}) \subseteq |\mathrm{co}|\big(D^{w^*}(X, u)\big).$$

As a consequence

$$B_{X_*} = \overline{|\mathrm{co}|}\big(D^{w^*}(X, u)\big).$$

We will deduce some consequences on the diameter two property.

Theorem 3.6. *Let X and Y be Banach spaces. Assume that $Z(X^{(\infty)})$ is infinite-dimensional and that there exists $f \in S_{Y^*}$ such that $n(Y^*, f) = 1$. Then the space $X \widehat{\otimes}_\pi Y$ has the diameter two property.*

Proof. Let W be a non-empty relatively weakly open subset of $B_{X\widehat{\otimes}_\pi Y}$. We can clearly assume that there is $\eta > 0$, $A_1, \ldots, A_m$ in $\mathcal{B}(X \times Y)$, and $z_0 \in S_{X\widehat{\otimes}_\pi Y}$ such that

$$W := \big\{z \in B_{X\widehat{\otimes}_\pi Y}\colon \big|\bar{A}_i(z) - \bar{A}_i(z_0)\big| < \eta,\ \forall 1 \leqslant i \leqslant m\big\}.$$

Since every weakly open set is norm open set, we can assume that

$$z_0 = \sum_{j=1}^{k} \alpha_j x_j \otimes y_j,$$

where $\alpha_j \in \mathbb{R}^+$ for every $1 \leqslant j \leqslant k$, $\sum_{j=1}^k \alpha_j = 1$, $x_j \in S_X$, and $y_j \in S_Y$, for all $j = 1, \ldots, k$. By Corollary 3.5, we have that $B_Y = \overline{|\mathrm{co}|}(D^{w^*}(Y^*, f))$. As a consequence, we can assume that y_j belongs to $D^{w^*}(Y^*, f)$ for every $j \in \{1, \ldots, k\}$. Let be $\varepsilon > 0$. Now we apply Lemma 2.5 to the Banach space X and the element $\sum_{j=1}^k \alpha_j x_j \otimes y_j$. There exist u_j, v_j in B_X for $1 \leqslant j \leqslant k$ and $\varphi \in B_{X^*}$ such that

$$z_1 = \sum_{j=1}^{k} \alpha_j u_j \otimes y_j, \qquad z_2 = \sum_{j=1}^{k} \alpha_j v_j \otimes y_j \in W$$

and $|\varphi(u_j) - 1| < \varepsilon$ and $|\varphi(v_j) + 1| < \varepsilon$ for all $j = 1, \dots, k$. We consider the bounded bilinear map $\phi : X \times Y \to \mathbb{K}$ given by $\phi(x, y) := \varphi(x) f(y)$. Then

$$\|z_1 - z_2\| \geqslant \left|\bar{\phi}(z_1 - z_2)\right| = \left|\sum_{j=1}^{k} \alpha_j \bigl(\varphi(u_j) - \varphi(v_j)\bigr) f(y_j)\right| \geqslant \sum_{j=1}^{k} \alpha_j (2 - 2\varepsilon) = 2 - 2\varepsilon.$$

We conclude that

$$\operatorname{diam} W \geqslant \|z_1 - z_2\| \geqslant 2 - 2\varepsilon,$$

foe every $\varepsilon > 0$, hence $\operatorname{diam} W = 2$. $\square$

We will provide some examples of spaces satisfying the assumption of the last statement. If μ is a σ-finite measure, then the space

- $Y = L_1(\mu)$, by taking the unit of $L_\infty(\mu)$ as f,
- $Y = L_1(\mu) \widehat{\otimes}_\pi L_\infty(\mu)$, so Y^* can be identified with $\mathcal{L}(L_\infty(\mu), L_\infty(\mu))$, and f is the identity operator on $L_\infty(\mu)$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$),
- in the complex space, $Y = L_1(\mu) \widehat{\otimes}_\pi L_\infty(\mu) \widehat{\otimes}_\pi L_\infty(\mu)$, so Y^* can be identified with the space of continuous bilinear mappings on $L_\infty(\mu) \times L_\infty(\mu)$ with values in $L_\infty(\mu)$, and f is the usual product on $L_\infty(\mu)$,

satisfies that $n(Y^*, f) = 1$. In the first and second cases it can be checked directly. Indeed the second example is a consequence of [12, Theorem 32.5]. The third one can be found in [22, Theorem 1.1]. Indeed, every CL-space (see [14] or [24] for the definitions) or more generally an almost CL-space satisfies that the unit sphere of its dual contains points where the numerical index is one. This class contains the spaces $L_1(\mu)$ and $\mathcal{C}(K)$. It is also known that $C(K, X)$ is an almost CL-space if X is an almost CL-space [24, Proposition 11].

Let us also observe that every Banach space has an equivalent norm for which the unit sphere of the dual has an element with numerical index one. If X is a Banach space, $x_0 \in S_X$ and M is a closed subspace of X such that $X = M \oplus \mathbb{K}x_0$, we consider the norm in X given by

$$|||m + \lambda x_0||| := \max\bigl\{\|m\|, |\lambda|\bigr\}.$$

So $X^* = M^* \oplus_1 \mathbb{K}x_0^*$ for some functional $x_0^* \in S_{X^*} \cap M^0$. Then it is immediate that for the norm $|||\ \ |||$ we have $n(X^*, x_0^*) = 1$.

4. A result for $C(K)$

On Theorems 2.6 and 3.6 we assumed conditions on both spaces in order to obtain the diameter two property for their projective tensor product. The next result shows that it is not needed any extra assumption in the case that one of the spaces is any infinite-dimensional $\mathcal{C}(K)$.

Theorem 4.1. *Let K be any infinite compact Hausdorff topological space and X a non-null Banach space. Then the space $Y := \mathcal{C}(K) \widehat{\otimes}_\pi X$ satisfies the diameter two property.*

Proof. Assume that W is a non-empty open set in (B_Y, w). Since K is infinite and $X \neq \{0\}$, then Y is infinite-dimensional and so $W \cap S_Y \neq \varnothing$. Since W is weakly open in B_Y, it is open in B_Y for the norm topology. Hence, for every $\varepsilon > 0$, there are $m \in \mathbb{N}$, $f_1, \dots, f_m \in S_{\mathcal{C}(K)}, x_1, \dots, x_m \in S_X$ and positive real numbers t_i $(1 \leqslant i \leqslant m)$ with $\sum_{i=1}^m t_i = 1$ such that

$$y_0 := \sum_{i=1}^{m} t_i f_i \otimes x_i \in W$$

and $\|y_0\| > 1 - \varepsilon^2$. Hence there is $y_0^* \in S_{Y^*}$ satisfying that $y_0^*(y_0) = \operatorname{Re} y_0^*(y_0) > 1 - \varepsilon^2$. Since Y^* is linearly isometric to $\mathcal{L}(\mathcal{C}(K), X^*)$, then there is $T_0 \in S_{\mathcal{L}(\mathcal{C}(K), X^*)}$ such that T_0 is the operator associated to the functional y_0^*. We consider the sets

$$G := \bigl\{i \in \{1, \dots, m\}:\ \operatorname{Re} T_0(f_i)(x_i) > 1 - \varepsilon\bigr\}, \qquad P := \{1, \dots, m\} \backslash G.$$

We know that

$$\begin{aligned} 1 - \varepsilon^2 < \operatorname{Re} y_0^*(y_0) &= \sum_{i=1}^{m} t_i \operatorname{Re} T_0(f_i)(x_i) = \sum_{i \in G} t_i \operatorname{Re} T_0(f_i)(x_i) + \sum_{i \in P} t_i \operatorname{Re} T_0(f_i)(x_i) \\ &\leqslant \sum_{i \in G} t_i + \sum_{i \in P} t_i (1 - \varepsilon) = 1 - \varepsilon \sum_{i \in P} t_i. \end{aligned}$$

Hence $\sum_{i\in P} t_i < \varepsilon$ and so

$$\sum_{i\in G} t_i > 1-\varepsilon. \tag{4.1}$$

Now we restrict the operator $T_0^{**}: \mathcal{C}(K)^{**} \to X^{***}$ to the linear space $\mathcal{B}(K)$ of bounded measurable functions on K, that clearly contains $\mathcal{C}(K)$. Let us remark that the restriction of the norm of $C(K)^{**}$ to $\mathcal{B}(K)$ is just the supremum norm, that is,

$$\|f\| := \sup_{t\in K} \bigl|f(t)\bigr| \quad \forall f \in \mathcal{B}(K).$$

Since the linear space of simple measurable functions on K is dense in $\mathcal{B}(K)$, there are $k \in \mathbb{N}$, $A_1, \dots, A_k \subset K$ measurable sets, non-empty and pairwise disjoint such that

$$\left\| \sum_{s=1}^{k} \beta_s^i \chi_{A_s} - f_i \right\| < \varepsilon \quad \forall 1 \leqslant i \leqslant m \tag{4.2}$$

for convenient scalars $\{\beta_s^i\colon 1 \leqslant i \leqslant m,\ 1 \leqslant s \leqslant k\}$ satisfying $|\beta_s^i| \leqslant 1$, $\forall i, s$. Since the subsets $\{A_s\colon 1 \leqslant s \leqslant k\}$ are non-empty and pairwise disjoint, the space M generated by $\{\chi_{A_s}\colon 1 \leqslant s \leqslant k\}$ is a subspace of $\mathcal{B}(K)$ is isometric to ℓ_∞^k. Indeed the unique linear mapping $\Psi : \ell_\infty^k \to M$ that satisfies

$$\Psi(e_s) = \chi_{A_s} \quad \text{for every } 1 \leqslant s \leqslant k \tag{4.3}$$

is a linear isometry.

We write

$$s_i := \sum_{s=1}^{k} \beta_s^i \chi_{A_s} \quad (1 \leqslant i \leqslant m). \tag{4.4}$$

We know that $s_i \in M$ and by (4.2) it holds that $\|s_i - f_i\| < \varepsilon$ for each i. Since it is satisfied that $\operatorname{Re} T(f_i)(x_i) > 1-\varepsilon$ for $i \in G$ and $\|T\| = 1$, then we deduce that

$$\operatorname{Re} T^{**}(s_i)(x_i) > 1 - 2\varepsilon \quad \forall i \in G. \tag{4.5}$$

Since K is infinite, there are sequences $\{g_n\}$ and $\{h_n\}$ in $\mathcal{C}(K)$ satisfying the conditions of Lemma 2.1.i) in [1]. That is, there are sequences of non-empty open subsets $\{U_n\}$ and $\{V_n\}$ of K such that

$$\begin{aligned} &\overline{V}_n \subset U_n \quad \forall n \in \mathbb{N}, \qquad U_n \cap U_m = \varnothing \quad \text{if } n \neq m, \\ &\operatorname{supp} h_n \subset V_n, \qquad \operatorname{supp} g_n \subset U_n \quad \forall n, \end{aligned} \tag{4.6}$$

and

$$g_{n|V_n} \equiv 1, \qquad 0 \leqslant g_n, h_n \leqslant 1\ \forall n, \qquad \|h_n\| = 1 \quad \forall n. \tag{4.7}$$

Since the functions $\{h_n\colon n \in \mathbb{N}\}$ have disjoint supports, then its linear span is isometric to c_0. Indeed there is a linear isometry from $[h_n\colon n \in \mathbb{N}]$ onto c_0 that maps $\{h_n\}$ into the usual Schauder basis of c_0. The same argument also holds for $\{g_n\colon n \in \mathbb{N}\}$. Then $\{g_n\}$ and $\{h_n\}$ are weakly null sequences in $\mathcal{C}(K)$ and so for each $1 \leqslant i \leqslant m$, the sequence $\{u_{i,n}\}$ given by

$$u_{i,n} := f_i \prod_{j=nk}^{(n+1)k-1} (1 - g_j) + \sum_{j=1}^{k} \beta_j^i h_{nk+j-1}$$

converges weakly to f_i in $\mathcal{C}(K)$. We will also check that $\|u_{i,n}\| \leqslant 1$ for each $1 \leqslant i \leqslant m$ and $n \in \mathbb{N}$. Let us fix n and i and choose $t \in K$. If $t \notin U_s$ for every $s \in \{j \in \mathbb{N}\colon nk \leqslant j \leqslant (n+1)k - 1\}$, then in view of (4.6) we have that $g_s(t) = h_s(t) = 0$ for every $nk \leqslant s \leqslant (n+1)k - 1$ and so $u_{i,n}(t) = f_i(t)$. Assume now that there is some $j_0 \in [1, k]$ such that $t \in U_{s_0}$, where $s_0 = nk + j_0 - 1$. If $t \in V_{s_0}$ then we have

$$u_{i,n}(t) = \beta_{j_0}^i h_{s_0}(t).$$

On the other hand, if $t \notin V_{s_0}$, then we obtain

$$u_{i,n}(t) = f_i(t)\bigl(1 - g_{s_0}(t)\bigr),$$

and in any case $|u_{i,n}(t)| \leqslant 1$. A similar argument proves that for every $1 \leqslant i \leqslant m$, the sequence $\{v_{i,n}\}$ given by

$$v_{i,n} := f_i \prod_{j=nk}^{(n+1)k-1} (1 - g_j) - \sum_{j=1}^{k} \beta_j^i h_{nk+j-1}$$

converges weakly to f_i and also belongs to the unit ball of $\mathcal{C}(K)$. Hence, for every $R \in \mathcal{L}(\mathcal{C}(K), X^*)$, it is satisfied that $\{R(u_{i,n})\}_n$ and $\{R(v_{i,n})\}_n$ converge weakly to $R(f_i)$ and so the above sequences converge also in the weak*-topology of X^* to $R(f_i)$. Hence for n large enough the elements

$$u := \sum_{i=1}^{m} t_i u_{i,n} \otimes x_i, \qquad v := \sum_{i=1}^{m} t_i v_{i,n} \otimes x_i$$

belong to W. Now we will estimate $\|u - v\|$. Notice that

$$u - v = \sum_{i=1}^{m} t_i (u_{i,n} - v_{i,n}) \otimes x_i = 2 \sum_{i=1}^{m} t_i \left(\sum_{j=1}^{k} \beta_j^i h_{nk+j-1} \right) \otimes x_i.$$

We write $Z := [h_j \colon nk \leqslant j \leqslant (n+1)k - 1]$, and we know that Z is isometric to ℓ_∞^k and it is 1-complemented in $\mathcal{C}(K)$. We denote by Φ the natural isometry from Z onto ℓ_∞^k. In order to check the last statement, in view of (4.7) and (4.6), we can choose $t_n \in V_n$ for every n satisfying $h_n(t_n) = 1$ and define the projection $P : \mathcal{C}(K) \to \mathcal{C}(K)$ by

$$P(f) = \sum_{s=nk}^{(n+1)k-1} f(t_s) h_s \quad \big(f \in \mathcal{C}(K)\big).$$

Since the open sets $\{V_n\}$ are pairwise disjoint and $\|h_n\| = 1$ for each n, then $\|P\| = 1$ and $P(h_s) = h_s$ for every $s \in [nk, (n+1)k - 1]$, so P is a norm-one projection onto Z. Now let us consider the linear isometry $L := \Psi \circ \Phi$ from Z onto M obtained as follows:

$$Z \xrightarrow{\Phi} \ell_\infty^k \xrightarrow{\Psi} M \subset \mathcal{B}(K) \subset \mathcal{C}(K)^{**},$$

hence $L(h_j) = \chi_{A_{j-nk+1}}$ for each $j \in [nk, (n+1)k - 1]$. We will denote by $R : X^{***} \to X^*$ the Dixmier projection.

For $1 \leqslant i \leqslant m$ in view of (4.4) we clearly have that

$$\big(R \circ T_0^{**} \circ L \circ P\big) \left(\sum_{j=1}^{k} \beta_j^i h_{nk+j-1} \right) = \big(R \circ T_0^{**} \circ L\big) \left(\sum_{j=1}^{k} \beta_j^i h_{nk+j-1} \right) = \big(R \circ T_0^{**}\big)(s_i).$$

Then the operator $S := R \circ T_0^{**} \circ L \circ P$ satisfies that $S \in \mathcal{L}(\mathcal{C}(K), X^*)$ and $\|S\| \leqslant \|T_0\| = 1$.

Hence

$$2 \geqslant \operatorname{diam} W \geqslant \|u - v\| \geqslant 2 \operatorname{Re} S \left(\sum_{i=1}^{m} t_i \left(\sum_{j=1}^{k} \beta_j^i h_{nk+j-1} \right) \right)(x_i) = 2 \sum_{i=1}^{m} t_i \operatorname{Re} T_0^{**}(s_i)(x_i),$$

$$2 \sum_{i \in G} t_i \operatorname{Re} T_0^{**}(s_i)(x_i) + 2 \sum_{i \in P} t_i \operatorname{Re} T_0^{**}(s_i)(x_i) \geqslant 2 \left(\sum_{i \in G} t_i (1 - 2\varepsilon) - \sum_{i \in P} t_i \right) \geqslant 2(1 - 2\varepsilon)(1 - \varepsilon) - 2\varepsilon.$$

Since ε is any positive real number, we deduce that $\operatorname{diam} W = 2$ as we wanted to show. $\square$

It is known that every space with the Daugavet property satisfies the diameter two property [28, Lemma 3]. In [20, Corollary 4.3] the authors provided an example of a two-dimensional complex normed space F such that $L_\infty^{\mathbb{C}}[0,1] \widehat{\otimes}_\pi F$ fails the Daugavet property. However our result can be applied to the previous space.

Besides $\mathcal{C}(K)$, $L_1(\mu)$ also satisfies the diameter two property if the measure μ is atomless. If we consider $L_1(\mu)$ (for an atomless measure μ) instead of $\mathcal{C}(K)$, Theorem 4.1 still holds. Indeed $L_1(\mu, X)$ does have the Daugavet property (see [21, Example, p. 858]) if μ is an atomless measure.

5. Injective tensor product

Until now along the paper we considered results stating the diameter two property for projective tensor products of Banach spaces. In the case of the injective tensor product, it will be enough to assume one restriction only to one of the spaces in order to obtain a positive result, as we will see later.

Lemma 5.1. *(See [10, Proposition 4.1].) Let X be a Banach space failing the diameter 2 property. Then $X^{(\infty}$ also fails this property.*

Lemma 5.2. *(See [10, Theorem 4.4].) Let X be a Banach space failing the diameter two property. Then there exists $m \in \mathbb{N}$ such that $\dim(Z(X^{(2n)})) \leqslant m$ for every $n \in \mathbb{N}$.*

If X and Y are Banach spaces over the same scalar field ($\mathbb{K}$), we recall that the *injective tensor product* of X and Y, denoted by $X \widehat{\otimes}_\varepsilon Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\| := \sup\left\{ \sum_{i=1}^n \left|x^*(x_i)y^*(y_i)\right|:\ u = \sum_{i=1}^n x_i \otimes y_i,\ n \in \mathbb{N},\ x_i \in X,\ y_i \in Y,\ \forall 1 \leqslant i \leqslant n,\ x^* \in S_{X^*},\ y^* \in S_{Y^*} \right\}.$$

Theorem 5.3. *Let X be a Banach space over $\mathbb{K}$, such that $\sup\{\dim Z(X^{(2n)}):\ n \in \mathbb{N}\} = \infty$. Then the space $X \widehat{\otimes}_\varepsilon Y$ satisfies the diameter two property, for every non-null Banach space Y.*

Proof. Since $X^{**}\widehat{\otimes}_\varepsilon Y$ can be seen as a subspace of $(X\widehat{\otimes}_\varepsilon Y)^{**}$ containing $X\widehat{\otimes}_\varepsilon Y$ [13, Lemma 1], we have that

$$(X\widehat{\otimes}_\varepsilon Y)^{**} \subseteq \left(X^{**}\widehat{\otimes}_\varepsilon Y\right)^{**} \subseteq (X\widehat{\otimes}_\varepsilon Y)^{(4}.$$

By applying again the mentioned result to $X^{**}\widehat{\otimes}_\varepsilon Y$, we obtain

$$X^{**}\widehat{\otimes}_\varepsilon Y \subseteq X^{(4}\widehat{\otimes}_\varepsilon Y \subseteq \left(X^{**}\widehat{\otimes}_\varepsilon Y\right)^{**}.$$

We conclude that

$$X\widehat{\otimes}_\varepsilon Y \subseteq X^{**}\widehat{\otimes}_\varepsilon Y \subseteq X^{(4}\widehat{\otimes}_\varepsilon Y \subseteq (X\widehat{\otimes}_\varepsilon Y)^{(4}.$$

By induction we prove that for every $n \in \mathbb{N}$

$$X\widehat{\otimes}_\varepsilon Y \subseteq X^{(2n}\widehat{\otimes}_\varepsilon Y \subseteq (X\widehat{\otimes}_\varepsilon Y)^{(2n}.$$

We fix $n \in \mathbb{N}$, then for $m \in \mathbb{N}$ we have that

$$(X\widehat{\otimes}_\varepsilon Y)^{(2m} \subseteq \left(X^{(2n}\widehat{\otimes}_\varepsilon Y\right)^{(2m} \subseteq \left((X\widehat{\otimes}_\varepsilon Y)^{(2n)}\right)^{(2m}.$$

This implies that

$$(X\widehat{\otimes}_\varepsilon Y)^{(\infty} \subseteq \left(X^{(2n}\widehat{\otimes}_\varepsilon Y\right)^{(\infty} \subseteq \left((X\widehat{\otimes}_\varepsilon Y)^{(2n)}\right)^{(\infty}.$$

For every Banach space Z and every $p \in \mathbb{N}$, we have $(Z^{(2p)})^{(\infty} = Z^{(\infty}$. It follows that

$$(X\widehat{\otimes}_\varepsilon Y)^{(\infty} = \left(X^{(2n}\widehat{\otimes}_\varepsilon Y\right)^{(\infty}.$$

By Lemma 5.1, if there is a relatively weakly open subset of $B_{X\widehat{\otimes}_\varepsilon Y}$ whose diameter is less than two, the same happens for the space $(X\widehat{\otimes}_\varepsilon Y)^{(\infty}$. By Lemma 5.2, there exists $m \in \mathbb{N}$ such that $\dim(Z((X\widehat{\otimes}_\varepsilon Y)^{(\infty)})) \leqslant m$. Hence, $\dim(Z((X^{(2n}\widehat{\otimes}_\varepsilon Y)^{(\infty)})) \leqslant m$ for all $n \in N$. As a consequence, $\dim(Z(X^{(2n}\widehat{\otimes}_\varepsilon Y)) \leqslant m$ for all $n \in N$. Since $Z(X^{(2n}\widehat{\otimes}_\varepsilon Y)$ contains a copy of $Z(X^{(2n)}) \otimes Z(Y)$ for all $n \in N$ (see [29] and also [11, pp. 129 and 171]) and $Z(Y) \neq \{0\}$, we conclude that $\dim(Z(X^{(2n)})) \leqslant m$ for all $n \in \mathbb{N}$. This contradicts the assumption. $\square$

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